

Notes on undecidability of Medvedev's logic

Prompter: Rodrigo Nicolau Almeida

Proof caretaker: Soren Knudstorp

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Abstract

We prove that Medvedev's intermediate logic is undecidable and is not recursively enumerable. The proof starts from the bounded canonical formulas of Zakharyashev in their algebraic presentation. For a finite subdirectly irreducible Heyting algebra A , with finite rooted dual poset P , failure of the canonical formula is reduced to the existence of an onto cofinal partial Esakia morphism from an entire Medvedev frame to P satisfying the selected closed-domain conditions. We then encode the periodic Wang domino problem into such finite partial morphisms. Four kinds of maximal vertices encode horizontal and vertical phases; guard chains force directed cycles; binary points encode successor and tile labels; and additional shield points distinguish incident from nonincident constraints. The converse direction is obtained from a periodic tiling by a calibrated support construction on a finite simplex. This gives a computable reduction from the periodic domino problem to non-theoremhood in Medvedev's logic.

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1 Methodology and Context

We here explain the methodology for the present working note. We explain as it stands the meaning of the two roles appearing: the “prompter” is the person who employed the Chat GPT tool to obtain the desired results, but whose expertise does not necessarily align completely with the problem - in this case, despite working in superintuitionistic logics, Almeida has never worked in issues of decidability of logics. His expertise is limited to knowing enough of the field and of the meaning of the relevant concepts to guide the model slightly towards the solution. The key motivation for prompting this question was in part the desire that his friend Soren Knudstorp not go through what he recently went through, namely, finding out that someone else had used a model to find a solution, and instead he be given the option of what to do with this proof and turn it into the best version it could possibly be.

On the other hand, Knudstorp's expertise, insight and alignment with the key issues of the problem are invaluable for the eventual solution, and so he has been deemed by Almeida as proof caretaker, should he accept the task of carefully working through this proof.

The prompt guidelines given were as follows: the model was instructed to employ Bezhanishvili and Zacharyashev's canonical formula axiomatization of superintuitionistic logics. After clarifying the precise criterion, the model was furthermore asked to obtain a simplification of this axiomatization by employing structural completeness and the disjunction property, structural hypotheses which might (Knudstorp will determine) have simplified the search space.

Subsequently the model was asked to execute a parallel search in several areas: graph combinatorics, CSP, finite topology combinatorics. It was tasked with creating several teams working towards a positive solution, as well as one team working towards a negative one, where it was specifically asked to employ the methods and ideas developed by Knudstorp in his recent work on undecidability of several logics. After two iterations the model produced a proof. The present note contains the summarized contents of that argument.

2 Introduction

For $n \geq 1$, the n -th Medvedev frame is

$$M_n = (\mathcal{P}^*([n]), \supseteq), \quad \mathcal{P}^*([n]) = \mathcal{P}([n]) \setminus \{\emptyset\}.$$

Thus $S \leq T$ in M_n precisely when $S \supseteq T$. Its least point is $[n]$, its maximal points are the singletons, and

$$\uparrow S = \{T : \emptyset \neq T \subseteq S\}.$$

Medvedev's logic is

$$\text{ML} = \bigcap_{n \geq 1} \text{Log}(M_n).$$

Equivalently, if $B_n = \text{Up}(M_n)$ denotes the Heyting algebra of upsets of M_n , then a formula belongs to ML exactly when it is valid in every B_n .

Medvedev's logic has the disjunction property and is structurally complete [10]. It is also known not to be finitely axiomatizable [8]. Recent accounts have continued to list its decidability, equivalently its recursive axiomatizability, as an open problem [4, 3]. The purpose of this note is to settle this problem negatively.

Theorem 2.1 (Main theorem). *The set of formulas valid in every Medvedev frame is co-recursively enumerable but not recursively enumerable. Consequently, Medvedev's logic is undecidable and has no recursively enumerable axiomatization.*

We obtain a second formulation in terms of the canonical axiomatization.

Theorem 2.2 (Canonical-pattern problem). *There is no algorithm which, given a finite rooted poset P and a finite set $D \subseteq \text{Up}(P)^2$, decides whether the bounded canonical formula $\rho(\text{Up}(P), D)$ belongs to ML. This set of positive canonical patterns is co-recursively enumerable but not recursively enumerable.*

The proof uses the doubly periodic domino problem. The key point is not merely to encode local tile compatibility. An onto partial morphism may use supports of arbitrary cardinality, and an arbitrary relation support can interact with a tile support without sharing the intended vertex. The guard and shield points introduced below rule out precisely these two evasions.

3 Canonical formulas and their dual semantics

3.1 Finite Heyting algebras and rooted duals

For a finite poset P , the set $\text{Up}(P)$ of upsets is a Heyting algebra. Meet and join are intersection and union, while

$$U \rightarrow V = P \setminus \downarrow (U \setminus V).$$

A finite Heyting algebra A is subdirectly irreducible exactly when $A \setminus \{1\}$ has a largest element s , called its opremum. Under finite Esakia duality, finite subdirectly irreducible Heyting algebras correspond to finite rooted posets, where *rooted* means having a least point. Indeed, if P has root r , then $P \setminus \{r\}$ is the largest proper upset of P .

3.2 The bounded canonical formula

Let A be a finite subdirectly irreducible Heyting algebra with opremum s , and let $D \subseteq A^2$ be arbitrary. Introduce a variable p_a for each $a \in A$, and put

$$\begin{aligned} \Delta(A, D) = & \bigwedge_{a,b \in A} (p_{a \wedge b} \leftrightarrow (p_a \wedge p_b)) \\ & \wedge \bigwedge_{a,b \in A} (p_{a \rightarrow b} \leftrightarrow (p_a \rightarrow p_b)) \\ & \wedge \bigwedge_{a \in A} (p_{\neg a} \leftrightarrow \neg p_a) \\ & \wedge \bigwedge_{(a,b) \in D} (p_{a \vee b} \leftrightarrow (p_a \vee p_b)). \end{aligned}$$

The bounded canonical formula associated with (A, D) is

$$\rho(A, D) = \Delta(A, D) \rightarrow p_s.$$

This is the formula denoted $\alpha(A, D)$ in [2, Definition 4.1]. It is equivalent to Zakharyashev's original relational canonical formula [2, Remark 5.13]. The adjective “bounded” is important: the negation diagram forces preservation of the bottom element and corresponds dually to cofinality. Some older conventions write an additional $\perp$ parameter for this version.

The set D is not required to satisfy a closure axiom. It records exactly the pairs whose joins must be preserved by the diagram map.

3.3 Partial Esakia morphisms

We recall the finite form of the generalized Esakia duality used for these formulas. Let X, Y be finite posets and let $f : X \rightarrow Y$ be a partial map. We write $f[\uparrow x]$ for the images of domain points in $\uparrow x$.

Definition 3.1. The map $f : X \rightarrow Y$ is a *partial Esakia morphism* when:

(PE1) if $x, z \in \text{dom}(f)$ and $x \leq z$, then $f(x) \leq f(z)$;

- (PE2) if $x \in \text{dom}(f)$ and $f(x) \leq y$, then some $z \in \text{dom}(f)$ satisfies $x \leq z$ and $f(z) = y$;
 (PE3) $x \in \text{dom}(f)$ exactly when $f[\uparrow x] = \uparrow y$ for some $y \in Y$;
 (PE4) $f[\uparrow x]$ is closed for every $x \in X$;
 (PE5) for each clopen upset $U \subseteq Y$, the set $X \setminus \downarrow f^{-1}(Y \setminus U)$ is a clopen upset of X .

It is *cofinal* when, for every $x \in X$, there is a $z \in \text{dom}(f)$ with $x \leq z$.

Finite Esakia spaces have the discrete topology. Thus the topological parts of (PE4) and (PE5) reduce to finiteness; the set in (PE5) is an upset because it is the complement of a downset.

Let P be the finite rooted dual of A , and write $\zeta(a) \subseteq P$ for the upset corresponding to $a \in A$. For upsets $U, V \subseteq P$, define

$$\mathcal{D}_{U,V} = \left\{ C \in \text{Ant}(P) : \begin{array}{l} C \subseteq U \cup V, \\ C \cap (U \setminus V) \neq \emptyset, \\ C \cap (V \setminus U) \neq \emptyset \end{array} \right\},$$

and put

$$\mathcal{D}_D = \bigcup_{(a,b) \in D} \mathcal{D}_{\zeta(a), \zeta(b)}.$$

Definition 3.2 (Zakharyashev closed-domain condition). A partial Esakia morphism $f : X \rightarrow P$ satisfies ZCDC($\mathcal{D}_D$) if

$$x \notin \text{dom}(f) \implies \min f[\uparrow x] \notin \mathcal{D}_D.$$

The following is the precise dual form of the canonical-formula lemma.

Theorem 3.3 (Canonical-formula criterion). *Let A be a finite subdirectly irreducible Heyting algebra, let P be its finite rooted dual, and let $D \subseteq A^2$. For every Esakia space X ,*

$$X \not\models \rho(A, D)$$

if and only if there are a closed upset $Y \subseteq X$ and an onto cofinal partial Esakia morphism

$$f : Y \rightarrow P$$

satisfying ZCDC($\mathcal{D}_D$).

This is [2, Theorem 5.10], based on the generalized Esakia duality and the closed-domain lemma [2, Lemma 5.9]. Notice the direction: the dual object P is an onto partial image of a closed upset of X . It is not embedded into X .

4 The whole-Medvedev-frame normalization

The special self-similarity of the frames M_n removes the closed upset from [theorem 3.3](#).

Lemma 4.1 (Principal-cone normalization). *Let P be a finite rooted poset with root r . Suppose that a closed upset $Y \subseteq M_N$ carries an onto cofinal partial Esakia morphism $f : Y \rightarrow P$ satisfying a given collection of ZCDC conditions. Then, for some $m \geq 1$, there is an onto cofinal partial Esakia morphism*

$$g : M_m \rightarrow P$$

from the whole ambient Medvedev frame, satisfying the same ZCDC conditions.

Proof. Choose $x \in \text{dom}(f)$ with $f(x) = r$, using onto-ness. Since Y is an upset of M_N , it contains $\uparrow x$. Because the order is reverse inclusion,

$$\uparrow x = \{z : \emptyset \neq z \subseteq x\} \cong M_{|x|}.$$

For every $z \in \uparrow x$, its upper cone computed inside $\uparrow x$ is the same as its upper cone in M_N . Consequently all five clauses of [definition 3.1](#), cofinality, and ZCDC are inherited by the restriction $g = f \upharpoonright \uparrow x$.

Finally, (PE2) at x gives

$$g[\uparrow x] = f[\uparrow x] = \uparrow f(x) = \uparrow r = P,$$

so the restriction is still onto. □

Define $E(P, D)$ to mean that, for some $n \geq 1$, there is an onto cofinal partial Esakia morphism $M_n \rightarrow P$ satisfying all the families $\mathcal{D}_{U,V}$ associated with D .

Corollary 4.2 (Exact Medvedev criterion). *For every finite rooted P and every $D \subseteq \text{Up}(P)^2$,*

$$\rho(\text{Up}(P), D) \notin \text{ML} \iff E(P, D).$$

Proof. If the formula is not in ML, it is refuted on some M_N . Apply [theorem 3.3](#) and then [lemma 4.1](#). Conversely, an $E(P, D)$ -witness is an instance of [theorem 3.3](#) with $Y = M_n$. □

Here “whole frame” refers to the ambient source M_n . The partial map need not have all of M_n as its domain.

4.1 Structural completeness and direct algebraic embeddings

Structural completeness gives a second proof of the normalization and a useful algebraic formulation.

Proposition 4.3 (Direct embedding form). *Let A be finite and subdirectly irreducible. Then*

$$\rho(A, D) \notin \text{ML}$$

if and only if, for some $n \geq 1$, there is an injective map $h : A \rightarrow B_n$ preserving $0, 1, \wedge, \rightarrow$ and satisfying

$$h(a \vee b) = h(a) \vee h(b) \quad ((a, b) \in D).$$

Proof. Write $\rho(A, D) = \Delta(A, D) \rightarrow p_s$. Suppose this formula is not in ML. By structural completeness, the rule

$$\frac{\Delta(A, D)}{p_s}$$

is not admissible. Thus some substitution σ satisfies

$$\sigma\Delta(A, D) \in \text{ML}, \quad \sigma p_s \notin \text{ML}.$$

Choose n and a valuation v in B_n such that $v(\sigma p_s) < 1$, and define

$$h(a) = v(\sigma p_a).$$

Because $\sigma\Delta(A, D)$ is valid in every Medvedev algebra, its diagram equations show that h preserves the stated operations and the selected joins.

It remains to prove injectivity. If $a \neq b$, assume without loss of generality that $a \not\leq b$. Then $c = a \rightarrow b < 1$, and since s is the opremum, $c \leq s$. If $h(a) = h(b)$, preservation of implication gives

$$h(c) = h(a) \rightarrow h(b) = 1.$$

On the other hand, monotonicity and $h(s) < 1$ give

$$1 = h(c) \leq h(s) < 1,$$

a contradiction. Hence h is injective.

Conversely, such an embedding, used as the valuation $p_a \mapsto h(a)$, makes the antecedent of $\rho(A, D)$ equal to 1 and its consequent $h(s) < 1$. It therefore refutes the formula in B_n . $\square$

The generalized Esakia dual of this direct embedding is precisely the whole-source partial surjection of [corollary 4.2](#). Structural completeness of each individual logic $\text{Log}(M_n)$, proved in [3, Proposition 4], yields the sharper same-index version.

The disjunction property is not used in this reduction. In particular, it does not permit the selected domain D to be erased. The selected joins will carry the local compatibility constraints of the tiling.

5 Face semantics on a finite simplex

We record an exact combinatorial form of partial Esakia morphisms from M_n . It explains both the relation with cover-preserving maps and why a fixed- n search is finite.

Let $f : M_n \rightarrow P$ be a partial Esakia morphism. For every nonempty $S \subseteq [n]$, put

$$A_f(S) = f[\{T \in \text{dom}(f) : \emptyset \neq T \subseteq S\}], \quad \lambda_f(S) = \min A_f(S).$$

Proposition 5.1 (Exact face recurrence). *The following hold.*

- (i) $A_f(S)$ is an upset of P .
- (ii) $S \in \text{dom}(f)$ exactly when $\lambda_f(S) = \{p\}$ for some p , in which case $f(S) = p$.

(iii) If $|S| > 1$, define

$$\mu_f(S) = \min \bigcup_{i \in S} \uparrow \lambda_f(S \setminus \{i\}).$$

Then either

$$\lambda_f(S) = \mu_f(S),$$

or

$$\lambda_f(S) = \{p\} \quad \text{and} \quad \text{Cov}(p) = \mu_f(S).$$

In the latter case S is a newly filled p -face.

(iv) If $S \notin \text{dom}(f)$, then $\lambda_f(S)$ is nonsingleton and ZCDC is exactly the requirement $\lambda_f(S) \notin \mathcal{D}_D$.

Proof. For (i), if $f(T) = p \in A_f(S)$ and $p \leq q$, clause (PE2) supplies a domain subface $R \subseteq T \subseteq S$ with $f(R) = q$.

If $S \in \text{dom}(f)$, then (PE3) gives $A_f(S) = \uparrow f(S)$, whose unique minimum is $f(S)$. Conversely, a finite upset with the unique minimum p is exactly $\uparrow p$, so (PE3) proves (ii).

Every proper subface of S is contained in a facet $S \setminus \{i\}$. Therefore the images of all proper domain subfaces form

$$B(S) = \bigcup_{i \in S} A_f(S \setminus \{i\}) = \bigcup_{i \in S} \uparrow \lambda_f(S \setminus \{i\}).$$

If S is not a newly filled face, then $A_f(S) = B(S)$, which gives the first alternative. If $f(S) = p \notin B(S)$, the back condition says that every strict successor of p occurs on a proper subface, while all images of proper subfaces lie above p . Hence $B(S) = \uparrow p \setminus \{p\}$, whose minimal points are exactly $\text{Cov}(p)$. This proves (iii). Part (iv) follows from (ii) and [definition 3.2](#). $\square$

For fixed n, P, D , the recurrence yields a finite constraint-satisfaction problem. Decidability of $E(P, D)$ would require an effective global bound on n . No such bound exists, as a consequence of the reduction below.

6 The periodic domino problem

A Wang tile t carries four colors

$$w(t), \quad e(t), \quad s(t), \quad n(t).$$

For a finite nonempty tile set W , a finite torus tiling is a map

$$\tau : \mathbb{Z}_m \times \mathbb{Z}_\ell \longrightarrow W$$

such that, with cyclic indices,

$$\begin{aligned} e(\tau(a, b)) &= w(\tau(a + 1, b)), \\ n(\tau(a, b)) &= s(\tau(a, b + 1)). \end{aligned}$$

The problem PER asks whether such positive m, ℓ and such a τ exist.

This is the doubly periodic domino problem proved undecidable by Gurevich and Koryakov [5]; Jeandel gives a modern proof in [6]. It is recursively enumerable: enumerate m, ℓ and the finitely many maps into W .

Some sources use “periodic” for the existence of only one nonzero period. For finite Wang shifts this gives the same existence problem. Quotienting by one period turns transverse rows into a bi-infinite walk in a finite graph; a directed cycle supplies a second period. Conversely, any rank-two period lattice contains nonzero axis-parallel periods, so the tiling factors through some $\mathbb{Z}_m \times \mathbb{Z}_\ell$.

Lemma 6.1 (Surjective range). *A finite tile set W has a finite torus tiling if and only if some nonempty $W' \subseteq W$ has a finite torus tiling whose range is all of W' .*

Proof. Given a W -tiling, take W' to be its finite nonempty range. The reverse implication follows from $W' \subseteq W$. $\square$

7 The finite canonical pattern associated with a tile set

Fix a nonempty finite tile set W . We construct a finite rooted poset P_W and a set $D_W \subseteq \text{Up}(P_W)^2$.

7.1 The poset

There are four maximal *role points*

$$H_0, H_1, V_0, V_1.$$

For every role X , add two points forming the chain

$$g_X < d_X < X.$$

Add horizontal relation points

$$R_{00}, R_{01}, R_{10}, \quad \text{Cov}(R_{ab}) = \{H_0, H_1\},$$

vertical relation points

$$S_{00}, S_{01}, S_{10}, \quad \text{Cov}(S_{ab}) = \{V_0, V_1\},$$

and, for every $i, j \in \{0, 1\}$ and $t \in W$, a tile point

$$T_{ij,t}, \quad \text{Cov}(T_{ij,t}) = \{H_i, V_j\}.$$

Finally, add a root r , whose covers are every g_X , every R_{ab} , every S_{ab} , and every $T_{ij,t}$. There are no other comparabilities. In particular, the relation and tile points are covers of the root, while

$$r < g_X < d_X < X.$$

The resulting P_W is finite and rooted. Therefore $A_W = \text{Up}(P_W)$ is a finite subdirectly irreducible Heyting algebra.

Write $P_W^\circ = P_W \setminus \{r\}$ and

$$\text{Max} = \{H_0, H_1, V_0, V_1\}.$$

7.2 Pair-filling cuts

For the six pairs

$$(H_0, H_1), \quad (V_0, V_1), \quad (H_i, V_j) \quad (i, j \in \{0, 1\}),$$

place the pair of upsets

$$U = \{X\}, \quad V = \{Y\}$$

in D_W . Since X, Y are distinct maximal points, $\mathcal{D}_{U,V}$ consists of the single antichain $\{X, Y\}$.

7.3 Serial cuts

Use the following data.

direction	target Y	nonoutputs N	output Q
$H_0 \rightarrow H_1$	H_1	$\{R_{00}, R_{10}\}$	R_{01}
$H_1 \rightarrow H_0$	H_0	$\{R_{00}, R_{01}\}$	R_{10}
$V_0 \rightarrow V_1$	V_1	$\{S_{00}, S_{10}\}$	S_{01}
$V_1 \rightarrow V_0$	V_0	$\{S_{00}, S_{01}\}$	S_{10}

For each row define

$$I = P_W^\circ \setminus (\{g_Y, Q\} \cup N), \quad U = I \cup \{g_Y\}, \quad V = I \cup N,$$

and place (U, V) in D_W . Thus

$$U \setminus V = \{g_Y\}, \quad V \setminus U = N, \quad P_W \setminus (U \cup V) = \{r, Q\}.$$

In particular $d_Y \in U \cap V$.

7.4 Compatibility cuts

Put $R_0 = R_{01}$ and $R_1 = R_{10}$. For $i, j \in \{0, 1\}$ and $t \in W$, let

$$G_{ij,t}^H = \{T_{1-i,j,t'} : e(t) = w(t')\}.$$

Define

$$\begin{aligned} I_{ij,t}^H &= P_W^\circ \setminus (\{R_i, T_{ij,t}, d_{H_i}, g_{H_i}\} \cup G_{ij,t}^H), \\ U_{ij,t}^H &= I_{ij,t}^H \cup \{R_i\}, \\ V_{ij,t}^H &= I_{ij,t}^H \cup \{T_{ij,t}\}. \end{aligned}$$

Place $(U_{ij,t}^H, V_{ij,t}^H)$ in D_W .

Likewise put $S_0 = S_{01}$, $S_1 = S_{10}$, and

$$G_{ij,t}^V = \{T_{i,1-j,t'} : n(t) = s(t')\}.$$

Set

$$\begin{aligned} I_{ij,t}^V &= P_W^\circ \setminus (\{S_j, T_{ij,t}, d_{V_j}, g_{V_j}\} \cup G_{ij,t}^V), \\ U_{ij,t}^V &= I_{ij,t}^V \cup \{S_j\}, \\ V_{ij,t}^V &= I_{ij,t}^V \cup \{T_{ij,t}\}, \end{aligned}$$

and add these pairs to D_W .

For a horizontal cut, the exclusive points are R_i and $T_{ij,t}$. The points outside the union are

$$\{r, d_{H_i}, g_{H_i}\} \cup G_{ij,t}^H.$$

The vertical statement is analogous.

Lemma 7.1 (Legitimacy of the domain). *Every set U, V used above is an upset of P_W . Hence D_W is a genuine finite subset of A_W^2 .*

Proof. Singletons of maximal points are upsets. For a serial cut, the elements removed from P_W° are g_Y and relation points covering the root. No point retained in I lies below one of them. Moreover, $d_Y, Y \in I$, so adjoining g_Y preserves upward closure, while the maximal successors of every point of N already belong to I .

For a compatibility cut, the binary relation and tile points that are removed all cover the root. The only additional removed chain points are g_X, d_X together. Thus no retained point lies below a removed one. Adjoining R_i or $T_{ij,t}$ preserves upward closure because their maximal successors remain in I . $\square$

8 Soundness: a partial morphism yields a torus

Assume $E(P_W, D_W)$, witnessed by an onto cofinal partial Esakia morphism

$$f : M_n \rightarrowtail P_W.$$

8.1 Roles of domain faces

Every singleton of M_n belongs to the domain. Indeed, cofinality applied to $\{a\}$ can only choose the nonempty subface $\{a\}$. Its image is a maximal point of P_W , because its upper cone in M_n contains only itself. We call a an X -vertex if $f(\{a\}) = X \in \mathbf{Max}$.

Lemma 8.1 (Support-role lemma). *If $F \in \text{dom}(f)$ and $f(F) = p$, then the set of role labels of the vertices of F is exactly*

$$\mathbf{Max}(\uparrow p).$$

In particular:

- (a) a face labelled g_X or d_X consists solely of X -vertices;
- (b) a face labelled R_{ab} contains only H_0, H_1 -vertices and contains both roles;
- (c) a face labelled S_{ab} has the analogous vertical property;

- (d) a face labelled $T_{ij,t}$ contains only H_i, V_j -vertices and contains both roles;
- (e) a face labelled r contains all four roles.

Proof. For every $a \in F$, the singleton $\{a\}$ is a domain subface, so its maximal label belongs to

$$f[\uparrow F] = \uparrow p.$$

Conversely, if X is a maximal point above p , (PE2) gives a domain subface $G \subseteq F$ labelled X . Since $f[\uparrow G] = \uparrow X = \{X\}$, every vertex of G has role X . Thus F contains an X -vertex. $\square$

8.2 Every mixed pair has a unique color

Lemma 8.2 (Pair filling). *Let x, y have one of the six designated role pairs*

$$(H_0, H_1), \quad (V_0, V_1), \quad (H_i, V_j).$$

Then $\{x, y\}$ is a domain face. It is labelled by exactly one of the corresponding R -, S -, or T -points.

Proof. If $\{x, y\}$ were outside the domain, the images of its proper nonempty subfaces would have frontier $\{X, Y\}$, forbidden by the pair-filling cut. Thus it is a domain face. If its label is p , then

$$f[\uparrow \{x, y\}] = \uparrow p.$$

The support-role lemma and the definition of P_W show that the only possible p is one of the designated binary points over X, Y . Uniqueness follows because f is a function. $\square$

We henceforth speak of the color of such a mixed edge.

8.3 The guards force directed cycles

Onto-ness supplies, for each role X , a domain face G_X with $f(G_X) = g_X$. By lemma 8.1, it consists only of X -vertices. Applying (PE2) first to d_X and then to X gives strictly nested domain faces

$$G_X \supsetneq D_X \supsetneq Z_X, \quad f(D_X) = d_X, \quad f(Z_X) = X.$$

Consequently $|G_X| \geq 3$.

Lemma 8.3 (Seriality). *For every $x \in G_{H_i}$, some $y \in G_{H_{1-i}}$ makes the edge $\{x, y\}$ have color R_i . Likewise, for every $x \in G_{V_j}$, some $y \in G_{V_{1-j}}$ makes $\{x, y\}$ have color S_j .*

Proof. We prove one of the four identical cases. Let $X \rightarrow Y$ be the selected direction, with output Q and nonoutputs N , and fix $x \in G_X$. Put

$$F = G_Y \cup \{x\}.$$

Suppose that every edge $\{x, y\}$, $y \in G_Y$, has a color in N .

First, Q cannot occur as the image of any domain subface of F . For if $K \subseteq F$ were labelled Q , the support-role lemma would give vertices of both roles in K . Their two-face is domain by lemma 8.2; its binary label belongs to $\uparrow Q$, and therefore equals Q , contrary to the assumption. The root r cannot occur either, because F has only the two roles X, Y , whereas an r -face contains all four roles.

The face F is not a domain face. Indeed, its image upset contains g_Y and at least one point of N . The only common lower bound of these two root covers is r , but an r -label has just been excluded. Consequently its frontier lies in

$$P_W \setminus \{r, Q\} = U \cup V.$$

Moreover g_Y is a minimal active point, since its only strict lower point is r , and every active nonoutput is minimal for the same reason. Thus the frontier meets both $U \setminus V = \{g_Y\}$ and $V \setminus U = N$. It belongs to $\mathcal{D}_{U,V}$, contradicting ZCDC. Hence an output edge exists. $\square$

Define a directed bipartite graph on $G_{H_0} \cup G_{H_1}$ by

$$x \longrightarrow y \iff \begin{cases} x \in G_{H_0}, y \in G_{H_1}, & \{x, y\} \text{ has color } R_{01}, \\ x \in G_{H_1}, y \in G_{H_0}, & \{x, y\} \text{ has color } R_{10}. \end{cases}$$

Every vertex has an outgoing edge, so finiteness yields a directed alternating cycle. The analogous construction yields a directed vertical cycle.

8.4 Compatibility on cycle products

Let $x \rightarrow y$ be an edge of the horizontal cycle, where x has role H_i , and let v be a vertex of the vertical cycle having role V_j . By pair filling, for unique tiles $t, t' \in W$,

$$f(\{x, v\}) = T_{ij,t}, \quad f(\{y, v\}) = T_{1-i,j,t'}.$$

Lemma 8.4 (Horizontal compatibility). *For the preceding data,*

$$e(t) = w(t').$$

Proof. The three two-faces of $F = \{x, y, v\}$ have the labels

$$R_i, \quad T_{ij,t}, \quad T_{1-i,j,t'}.$$

The face F is not in the domain. Its image on proper subfaces has three distinct minimal root covers, and their only common lower bound is r . But an r -face must contain all four roles, while F contains only H_0, H_1, V_j . Since every proper nonempty subface is either a singleton or one of the three mixed pairs, the frontier is exactly

$$\{R_i, T_{ij,t}, T_{1-i,j,t'}\}.$$

If t' were horizontally incompatible with t , its target point would belong to $I_{ij,t}^H$. The displayed frontier would then be contained in $U_{ij,t}^H \cup V_{ij,t}^H$ and would meet the two exclusive singleton parts R_i and $T_{ij,t}$. This contradicts ZCDC. $\square$

The vertical compatibility lemma is identical, giving $n(t) = s(t')$ along every vertical cycle edge.

Index the horizontal and vertical directed cycles cyclically. Assign to a pair of cycle vertices (x, v) the unique tile t for which the mixed edge $\{x, v\}$ has label $T_{ij,t}$, with i, j determined by the two roles. The horizontal and vertical compatibility lemmas show that this is a finite torus tiling by W . We have proved:

Proposition 8.5 (Soundness). *If $E(P_W, D_W)$, then W has a finite torus tiling.*

9 Completeness: a torus yields a partial morphism

Suppose now that

$$\tau : \mathbb{Z}_m \times \mathbb{Z}_\ell \longrightarrow W$$

is a finite torus tiling whose range is all of W . Put

$$M = 3m, \quad L = 3\ell,$$

and extend τ periodically to $\mathbb{Z}_M \times \mathbb{Z}_L$.

Take four pairwise disjoint coordinate blocks

$$H_i = \{h_i(a) : a \in \mathbb{Z}_M\}, \quad V_j = \{v_j(b) : b \in \mathbb{Z}_L\},$$

and let

$$\Omega = H_0 \sqcup H_1 \sqcup V_0 \sqcup V_1.$$

We shall define the partial morphism on $M_\Omega = (\mathcal{P}^*(\Omega), \supseteq)$.

9.1 Calibrated supports

For each $p \in P_W$, define a nonempty family $\Sigma_p \subseteq \mathcal{P}^*(\Omega)$ as follows.

- (S1) For a role X , every singleton from the block X belongs to Σ_X .
- (S2) Every two-element subset of a role block X belongs to Σ_{d_X} .
- (S3) The whole block X is the unique member of Σ_{g_X} .
- (S4) For $i \in \{0, 1\}$ and $a \in \mathbb{Z}_M$, put

$$\{h_i(a), h_{1-i}(a+1)\} \in \Sigma_{R_i}.$$

Every remaining H_0 - H_1 pair belongs to $\Sigma_{R_{00}}$.

- (S5) For $j \in \{0, 1\}$ and $b \in \mathbb{Z}_L$, put

$$\{v_j(b), v_{1-j}(b+1)\} \in \Sigma_{S_j}.$$

Every remaining V_0 - V_1 pair belongs to $\Sigma_{S_{00}}$.

- (S6) For every i, j, a, b , put

$$\{h_i(a), v_j(b)\} \in \Sigma_{T_{ij, \tau(a \bmod m, b \bmod \ell)}}.$$

(S7) Put $\Omega \in \Sigma_r$.

Because $M, L \geq 3$, the two directed successor matchings in each pair of phase blocks are disjoint and default R_{00} - and S_{00} -pairs exist. Since τ is onto W , every tile point has a support in each of its four phases. Hence every $p \in P_W$ has a support.

For a nonempty $S \subseteq \Omega$, define

$$\text{Act}(S) = \{p \in P_W : \text{some } H \in \Sigma_p \text{ satisfies } H \subseteq S\}.$$

Lemma 9.1 (Calibration). *For every $p \in P_W$ and every $H \in \Sigma_p$,*

$$\text{Act}(H) = \uparrow p.$$

Proof. The verification follows directly from the types of supports.

- A singleton role support activates exactly its maximal role.
- A same-role pair activates exactly d_X and X . It does not contain the whole role block, since block size is at least three.
- A whole role block activates exactly g_X, d_X, X .
- A horizontal relation support activates its unique relation label and the two horizontal roles; a vertical relation support is analogous.
- A cell support activates its unique tile point and its horizontal and vertical roles.
- The full set Ω activates every point of P_W , and hence exactly $\uparrow r = P_W$.

No support of another type fits into any of the proper supports listed above. These sets are precisely the principal upsets determined by the definition of P_W . $\square$

We use a general support lemma.

Lemma 9.2 (Support realization). *Let P and V be finite. Suppose nonempty support families $\Sigma_p \subseteq \mathcal{P}^*(V)$ satisfy*

$$\{q : \exists K \in \Sigma_q (K \subseteq H)\} = \uparrow p \quad (H \in \Sigma_p).$$

For nonempty $S \subseteq V$, define $\text{Act}(S)$ by support containment, and define a partial map by

$$S \in \text{dom}(f), f(S) = p \iff \text{Act}(S) = \uparrow p.$$

Then $f : M_V \rightarrow P$ is an onto partial Esakia morphism. If every singleton is a calibrated support, it is cofinal.

Proof. First, $\text{Act}(S)$ is an upset. If a support $H \subseteq S$ activates $p \leq q$, calibration of H supplies a q -support contained in H , and hence in S .

The central identity is

$$f[\uparrow S] = \text{Act}(S). \tag{1}$$

For the left-to-right inclusion, if $T \subseteq S$ is a domain face with $f(T) = p$, then $p \in \text{Act}(T) \subseteq \text{Act}(S)$. Conversely, if $p \in \text{Act}(S)$, choose $H \in \Sigma_p$ with $H \subseteq S$. Calibration gives $\text{Act}(H) = \uparrow p$, so $H \in \text{dom}(f)$ and $f(H) = p$. Thus $p \in f[\uparrow S]$.

Now suppose $S, T \in \text{dom}(f)$ and $S \leq T$, which means $T \subseteq S$. Then

$$\uparrow f(T) = \text{Act}(T) \subseteq \text{Act}(S) = \uparrow f(S),$$

so $f(S) \leq f(T)$, proving (PE1). If $f(S) = p \leq q$, the identity (1) supplies a domain subface $H \subseteq S$ with $f(H) = q$, proving (PE2). Clause (PE3) is the domain definition together with (1). Clause (PE4) holds by finiteness.

For an upset $U \subseteq P$,

$$M_V \setminus \downarrow f^{-1}(P \setminus U) = \{S : \text{Act}(S) \subseteq U\}.$$

This is an upset of M_V , since $S \leq T$ implies $\text{Act}(T) \subseteq \text{Act}(S)$, and it is clopen by finiteness. This proves (PE5).

Every p is attained on each member of Σ_p , proving onto-ness. If every singleton is a domain face, then every nonempty S has a singleton subface T , so $S \leq T \in \text{dom}(f)$; this is cofinality. $\square$

Apply lemma 9.2 using lemma 9.1. We obtain an onto cofinal partial Esakia morphism

$$f : M_\Omega \rightarrow P_W.$$

9.2 The global ZCDC verification

We must check all non-domain faces, not only the displayed supports.

Lemma 9.3 (Outside-point principle). *Let $U, V \subseteq P_W$ be upsets and $S \subseteq \Omega$. If some $q \in \text{Act}(S)$ lies outside $U \cup V$, then $\min \text{Act}(S)$ has a point outside $U \cup V$.*

Proof. Choose a minimal active $p \leq q$. The complement of the upset $U \cup V$ is a downset. Since q is in this complement and $p \leq q$, p is also in the complement. $\square$

Let $S \notin \text{dom}(f)$ and put $C = \min \text{Act}(S)$.

For a pair-filling cut, suppose $C \in \mathcal{D}_{\{X\}, \{Y\}}$. Then $C = \{X, Y\}$, so S contains vertices of both roles. Their mixed pair has a relation or tile support q below X, Y , and $q \notin \{X, Y\}$. The outside-point principle contradicts $C \subseteq \{X, Y\}$.

For a serial cut, suppose $C \in \mathcal{D}_{U, V}$. Since the exclusive parts are $\{g_Y\}$ and N , the points g_Y and some $q \in N$ are active. Activation of g_Y means that the entire target block Y is contained in S . A support for $q \in N$ contains a source-role vertex x . The prescribed successor of x in the complete target block gives a Q -support contained in S . Since $Q \notin U \cup V$, the outside-point principle contradicts the assumed forbidden frontier.

For a horizontal compatibility cut, a forbidden frontier activates both R_i and $T_{ij,t}$. Choose supports

$$\{a, b\} \in \Sigma_{R_i}, \quad \{c, v\} \in \Sigma_{T_{ij,t}},$$

where $a, c \in H_i$, $b \in H_{1-i}$, and $v \in V_j$.

If $a \neq c$, the same-role pair $\{a, c\}$ activates d_{H_i} , which lies outside the compatibility union. If $a = c$, then b is the prescribed horizontal successor of a . The pair $\{b, v\}$ activates

$$T_{1-i,j,t'} \in G_{ij,t}^H,$$

because the original τ satisfies horizontal compatibility. This point also lies outside the compatibility union. Both cases contradict the outside-point principle. The vertical compatibility cuts are identical, using d_{V_j} and the north-south condition.

We have checked every selected pair and every non-domain S , so f satisfies ZCDC. Therefore:

Proposition 9.4 (Completeness). *If W has a finite torus tiling using every tile of W , then $E(P_W, D_W)$.*

10 The undecidability reduction

For each nonempty $W' \subseteq W$, form the finite rooted poset $P_{W'}$, the finite domain $D_{W'}$, and the canonical formula

$$\rho_{W'} = \rho(\text{Up}(P_{W'}), D_{W'}).$$

Theorem 10.1 (Tiling equivalence). *For every finite nonempty Wang tile set W ,*

$$W \in \text{PER} \iff \exists \emptyset \neq W' \subseteq W \quad \rho_{W'} \notin \text{ML}.$$

Proof. Suppose first that $W \in \text{PER}$. By lemma 6.1, some nonempty $W' \subseteq W$ has a surjective finite torus tiling. Proposition 9.4 gives $E(P_{W'}, D_{W'})$, and corollary 4.2 gives $\rho_{W'} \notin \text{ML}$.

Conversely, if $\rho_{W'} \notin \text{ML}$, then corollary 4.2 yields $E(P_{W'}, D_{W'})$. Proposition 8.5 gives a finite torus tiling by W' , which is also a tiling by W . $\square$

Rename variables between different conjuncts if necessary, and define the single formula

$$\Phi_W = \bigwedge_{\emptyset \neq W' \subseteq W} \rho_{W'}.$$

This construction is effective. Its exponential size is irrelevant to computability.

Corollary 10.2 (Many-one reduction). *For every finite nonempty W ,*

$$W \in \text{PER} \iff \Phi_W \notin \text{ML}.$$

Consequently non-theoremhood in ML, and hence theoremhood in ML, is undecidable.

Proof. A conjunction belongs to a logic exactly when every conjunct belongs to the logic. Apply theorem 10.1. Since $W \mapsto \Phi_W$ is a total computable map and PER is undecidable, the conclusion follows. $\square$

For the problem restricted to one canonical formula, the same construction is a finite nonadaptive truth-table reduction: query the finitely many patterns $\rho_{W'}$ and accept exactly when at least one is a non-theorem. We do not claim that Φ_W itself is a single canonical formula.

11 Computability-theoretic consequences

Proof of theorem 2.1. Nonmembership in ML is recursively enumerable. Given a formula φ , enumerate $n = 1, 2, \dots$ and, for each n , the finitely many valuations of the variables of φ in the finite algebra B_n . This process halts exactly when it finds a refutation. Thus ML is co-recursively enumerable.

By corollary 10.2, ML is undecidable. If ML were also recursively enumerable, dovetailing enumerations of membership and nonmembership would decide it. Hence ML is not recursively enumerable.

The deductive closure of any recursively enumerable set of axioms under the usual effective finitary intuitionistic calculus is recursively enumerable. It follows that ML has no recursively enumerable axiomatization. $\square$

Let

$$\text{Can}(\text{ML}) = \{\langle P, D \rangle : P \text{ is finite rooted, } D \subseteq \text{Up}(P)^2, \rho(\text{Up}(P), D) \in \text{ML}\}.$$

Proof of theorem 2.2. The complement of $\text{Can}(\text{ML})$ is recursively enumerable by the same finite-countermodel search. Undecidability follows from the finite truth-table reduction in theorem 10.1.

Finally, the canonical-formula axiomatization theorem says that every intermediate logic, in particular ML, is axiomatized by a subset of the canonical formulas it contains [2, Theorem 4.9]. Consequently all of $\text{Can}(\text{ML})$ also axiomatize ML. If $\text{Can}(\text{ML})$ were recursively enumerable, this would give a recursively enumerable axiomatization of ML, contradicting theorem 2.1. Hence it is co-recursively enumerable but not recursively enumerable. $\square$

Corollary 11.1 (No computable dimension bound). *There is no total computable function $b(P, D)$ with the property that whenever $E(P, D)$ holds, it has a witness on some M_n with $n \leq b(P, D)$.*

Proof. Such a function would decide $E(P, D)$: compute the bound and exhaustively test the finitely many partial maps on $M_1, \dots, M_{b(P, D)}$. Applying this procedure to all nonempty $W' \subseteq W$ would decide PER, by theorem 10.1. $\square$

Thus no general folding, finite-profile, or terminating support-chase theorem can yield an effective cutoff for the canonical-pattern problem. Fixed dimensions remain finite CSPs, but the least required dimension encodes unbounded periodic-tiling information.

12 Relation to the motivating search

The canonical-formula analysis produces a genuine simplification before the undecidability construction begins:

1. the arbitrary closed upset in the standard dual criterion can be replaced by a whole Medvedev frame, using lemma 4.1;
2. structural completeness replaces the algebraic homomorphic image by a direct bounded implicative-semilattice embedding into B_n , using proposition 4.3;

3. the resulting fixed- n search has the exact cover recurrence of [proposition 5.1](#).

What cannot be simplified away is the selected join domain D . The pair-filling, serial, and compatibility constraints in D_W are exactly what makes it possible to express a two-dimensional computation. The disjunction property does not constrain these local closed-domain conditions.

The construction was suggested in part by the use of powerset semilattices in recent tiling encodings for relevant and substructural logics, especially Knudstorp's work [7]. A direct transplantation does not respect persistence and has the wrong semidecidability orientation for the present problem. Passing to finite torus tilings, forcing cycles by guard faces, and adding the d_X -shields resolves those obstructions.

13 Verification notes

The proof separates the points at which an arbitrary witness could otherwise evade the intended local encoding.

- The support-role lemma makes no minimal-support assumption. It follows directly from the p-back condition.
- Pair filling proves that every active binary label on a relevant face is already visible on an actual two-face.
- The serial cuts use saturated intersections: every irrelevant nonroot label lies in $U \cap V$, so the only possible outside repairs are the intended successor and the root.
- On a compatibility triangle there is only one vertex of the source role, so neither d_X nor g_X can occur, and the missing fourth role excludes the root.
- In the converse construction, the support-realization identity $f[\uparrow S] = \text{Act}(S)$ verifies the exact-domain condition on every face, not merely on the displayed supports.
- The outside-point principle checks all larger forbidden antichains, rather than only their smallest examples.

Finite exhaustive checks were also performed on two constructed witnesses. For a three-tile cyclic 3×3 torus, the validator reported 31 target points, 12 coordinates, 4095 nonempty faces, 83 domain faces, 34 cuts, and zero failures. For a two-tile checkerboard 4×4 torus, it reported 27 target points, 16 coordinates, 65535 nonempty faces, 157 domain faces, 26 cuts, and zero failures. These computations are corroborative; the proof above treats arbitrary tile sets and arbitrary witness dimensions.

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